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1995 J. Phys. A: Math. Gen. 28 6915

(http://iopscience.iop.org/0305-4470/28/23/031)

J. Phys. A: Math. Gen. 28 (1995) 6915–6930. Printed in the UK

Integral transforms for conformal field theories with a boundary

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Received 11 July 1995

Abstract. A new method is developed for solving the conformally invariant integrals that arise in conformal field theories with a boundary. The presence of a boundary makes previous techniques for theories without a boundary less suitable. The method makes essential use of an invertible integral transform, related to the radon transform, involving integration over planes parallel to the boundary. For successful application of this method several non-trivial hypergeometric function relations are also derived.

1. Introduction

At a critical point most statistical mechanical systems are not only scale invariant but are also conformally invariant [1, 2]. This principle has profound implications for calculations of the correlation functions, critical exponents and universal amplitudes of such systems [3]. In two dimensions, where the conformal group is infinite dimensional, multipoint correlation functions are more strongly constrained then in dimension $d > 2$, where the conformal group is finite. However, consideration of $d > 2$ is also important, particularly in the statistical mechanical context when $d = 3$. In the case of general d conformal invariance still provides quite powerful constraints. For example, in the infinite geometry $\mathbb{R}^d$ the forms of the two and three point functions of scalar fields in a conformal field theory are determined exactly (up to normalization) by the restrictions of conformal invariance.

Cardy has shown how to generalize the principle of conformal invariance to the case of the semi-infinite geometry $\mathbb{R}^d_+$, so that surface critical phenomena can be probed using these techniques [1, 4]. In $\mathbb{R}^d_+$ it is only appropriate to have conformal invariance under conformal transformation which leave the boundary fixed. In this case the restrictions on the form of correlations functions are not as strong. In particular the form of the two-point function of a scalar field in $\mathbb{R}^d_+$ is restricted by conformal invariance only up to some function of a single conformally invariant variable [1]. This function must be then be determined for the particular theory under consideration.

In this paper we outline a powerful method, which makes essential use of conformal invariance, for calculating the two-point functions of scalar, vector and tensor fields of conformal field theories in the semi-infinite space $\mathbb{R}^d_+$. In particular we give a prescription for treating the conformally invariant integrals that arise in a diagrammatic expansion of the theory. Techniques for handling such integrals have been developed for the infinite space

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$\mathbb{R}^d$, and have proven to be very useful [5, 6]. However these techniques do not extend to $\mathbb{R}^d_+$ and so this alternative technique is developed.

2. Conformal invariance

A transformation of coordinates $x_\mu \to x^g_\mu(x)$ is a conformal transformation if it leaves the line element unchanged up to a local scale factor $\Omega(x)$. That is

$$\mathrm{d}x^g_\mu \mathrm{d}x^g_\mu = \Omega(x)^{-2} \mathrm{d}x_\mu \mathrm{d}x_\mu . \tag{2.1}$$

For the discussion of two-point functions of fields in a conformal field theory we need to consider the effect of conformal transformations on these fields. If a field $\mathcal{O}(x)$ transforms under the conformal group as

$$\mathcal{O}(x) \to \mathcal{O}^g(x^g) = \Omega(x)^\eta \mathcal{O}(x) \tag{2.2}$$

for some η, then $\mathcal{O}(x)$ is said to be a quasi-primary scalar field with scale dimension η. A quasi-primary vector field $\mathcal{V}_\mu(x)$ with scale dimension η is one which transforms as

$$\mathcal{V}(x) \to \mathcal{V}^g_\mu(x^g) = \Omega(x)^\eta \mathcal{R}_{\mu\alpha}(x) \mathcal{V}_\alpha(x) \tag{2.3}$$

where $\mathcal{R}_{\mu\alpha}(x) = \Omega(x)\partial x^g_\mu / \partial x_\alpha$. The transformation for quasi-primary tensor fields follows analogously. We will restrict our attention to quasi-primary fields in this paper.

In the semi-infinite space $\mathbb{R}^d_+$ we define coordinates $x_\mu = (y, \boldsymbol{x})$ where y measures the perpendicular distance from the boundary, and the $\boldsymbol{x}_i$ are coordinates in the $(d-1)$-dimensional hyperplanes parallel to the boundary. The two-point functions of scalar operators are restricted by translational and rotational invariance in planes parallel to the boundary to be

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(x')\rangle = G(y, y', |\boldsymbol{x} - \boldsymbol{x}'|) \tag{2.4}$$

and scale invariance further restricts the form of G to depend on two independent scale invariant variables s^2/y^2 and s^2/y'^2, where $s^2 = (x - x')^2$. This situation should be contrasted with the case of infinite space where it is not possible to construct a variable from two points which is invariant under all of scale, translational and rotational transformations.

For two points in $\mathbb{R}^d_+$ conformal invariance provides further restrictions. Under conformal transformations which leave the boundary fixed

$$s^2 \to \frac{s^2}{\Omega(x)\Omega(x')} \qquad y \to \frac{y}{\Omega(x)} \qquad y' \to \frac{y'}{\Omega(x')} \tag{2.5}$$

so that only one independent conformally invariant variable can be constructed from two points

$$\xi = \frac{s^2}{4yy'} \qquad \text{or} \qquad v^2 = \frac{s^2}{\bar{s}^2} = \frac{\xi}{1+\xi} \tag{2.6}$$

where $\bar{s}^2 = (\boldsymbol{x} - \boldsymbol{x}')^2 + (y + y')^2$ is the square of the distance along the path between x and the image point of x'.

As a consequence, the correlation function of two quasi-primary scalar fields may be written as

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(x')\rangle = \frac{1}{(2y)^{\eta_1}} \frac{1}{(2y')^{\eta_2}} f(\xi) \tag{2.7}$$

for some arbitrary function $f(\xi)$†.

† The $\xi \to 0$ and $\xi \to \infty$ limiting behaviour of this function is fixed by the operator product and boundary operator expansions [7].

As an example we consider free scalar field theory, where the field $\phi(x)$ satisfies Dirichlet or Neumann boundary conditions at $y = 0$. Then by the method of images the Green function is simply

$$\langle\phi(x)\phi(x')\rangle = G_\phi(x,x') = A\left(\frac{1}{s^{d-2}} \pm \frac{1}{\bar{s}^{d-2}}\right) = \frac{A}{(4yy')^{\eta_\phi}} f_\phi(\xi) \qquad (2.8)$$

where

$$A = \frac{1}{(d-2)S_d} \qquad \eta_\phi = \tfrac{1}{2}d - 1 \qquad f_\phi(\xi) = \xi^{-\eta_\phi} \pm (1+\xi)^{-\eta_\phi}. \qquad (2.9)$$

In the above expression the upper (lower) sign corresponds to Neumann (Dirichlet) boundary conditions and the factor $S_d = 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$ is the area of a unit hypersphere in d dimensions.

In [7], henceforth referred to as I, the form of the two-point functions of scalar, vector and tensor fields was worked out in detail for the $O(N)$ sigma model in both the ε and large-N expansions. These calculation were significantly simplified by the use of a new technique to solve the conformally invariant integrals on $\mathbb{R}^d_+$ that naturally arise. In the next section this technique is discussed in detail.

3. Parallel transform method

We consider integrals of the form

$$\begin{aligned} f(\xi) &= \int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}\boldsymbol{r}\, \frac{1}{(2z)^d} f_1(\tilde{\xi}) f_2(\tilde{\xi}') \\ \tilde{\xi} &= \frac{(x-r)^2}{4yz} \qquad \tilde{\xi}' = \frac{(x'-r)^2}{4y'z} \qquad r = (z, \boldsymbol{r}) \end{aligned} \qquad (3.1)$$

where conformal invariance restricts the form of the integral to be a function of ξ only. This follows because under conformal transformations which leave the boundary fixed, the integration measure transforms as $\mathrm{d}^d x \to \Omega(x)^{-d}\mathrm{d}^d x$ and the factor $1/(2z)^d \to \Omega(x)^d/(2z)^d$ so the local scaling factor cancels.

Given functions f_1 and f_2 we may solve integrals of this type indirectly by first integrating $f(\xi)$ over hyperplanes parallel to the boundary†

$$\int \mathrm{d}^{d-1}\boldsymbol{x}\; f(\xi) = (4yy')^\lambda \hat{f}(\rho) \qquad \rho = \frac{(y-y')^2}{4yy'} \qquad \lambda = \tfrac{1}{2}(d-1) \qquad (3.2)$$

which defines the function $\hat{f}(\rho)$ *to be*

$$\hat{f}(\rho) = \frac{\pi^\lambda}{\Gamma(\lambda)} \int_0^\infty \mathrm{d}u\; u^{\lambda-1} f(u+\rho). \qquad (3.3)$$

The crucial point is that this defines an integral transform $f \to \hat{f}$ which is invertible. Thus $f(\xi)$ can be retrieved from $\hat{f}(\rho)$ via

$$f(\xi) = \frac{1}{\pi^\lambda\Gamma(-\lambda)} \int_0^\infty \mathrm{d}\rho\; \rho^{-\lambda-1} \hat{f}(\rho+\xi). \qquad (3.4)$$

The integral in the above formula is actually singular for values of λ that we consider here, but the inversion formula may still be defined by analytic continuation in λ from $Re(\lambda) < 0$.

† This is related to the Radon transformation of $f(x)$ [8], which is defined as the integral of $f(x)$ over all possible hyperplanes in $\mathbb{R}^d$. Here we consider integrals over the subset of hyperplanes in $\mathbb{R}^d_+$ which are parallel to the boundary.

To verify that the transformation (3.3) is compatible with the inversion formula (3.4) it is sufficient to make use of the following relation involving generalized functions:

$$\int du\ (\rho-u)_+^{\mu-1}u_+^{\lambda-1} = B(\mu,\lambda)\,\rho_+^{\mu+\lambda-1} \sim \Gamma(-\lambda)\Gamma(\lambda)\delta(\rho) \qquad \text{as}\quad \mu\to-\lambda\,. \tag{3.5}$$

For the case $d=3$ when $\lambda=1$ we use

$$\frac{\rho_+^{-\lambda-1}}{\Gamma(-\lambda)} \sim \delta'(\rho) \qquad \text{as}\quad \lambda\to 1 \tag{3.6}$$

to reduce the inversion formula (3.4) to the simple form

$$f(\xi) = -\frac{1}{\pi}\,\hat{f}'(\xi)\,. \tag{3.7}$$

Now that this parallel transform has been defined it is possible to derive an integral relation for the transformed functions by integrating $f(\xi)$ in (3.1) with respect to x so that

$$\hat{f}(\rho) = \int_0^\infty dz\ \frac{1}{2z}\hat{f}_1(\tilde{\rho})\hat{f}_2(\tilde{\rho}') \qquad \tilde{\rho} = \frac{(y-z)^2}{4yz} \qquad \tilde{\rho}' = \frac{(y'-z)^2}{4y'z}\,. \tag{3.8}$$

In order to solve integrals of this type we first change variables $z = e^{2\theta}$, $y = e^{2\theta_1}$ and $y' = e^{2\theta_2}$ so that equation (3.8) becomes

$$\hat{f}\left(\sinh^2(\theta_1-\theta_2)\right) = \int_{-\infty}^{\infty} d\theta\ \hat{f}_1\left(\sinh^2(\theta-\theta_1)\right)\hat{f}_2\left(\sinh^2(\theta-\theta_2)\right)\,. \tag{3.9}$$

Now by taking the Fourier transform

$$\tilde{\hat{f}}(k) = \int_{-\infty}^{\infty} d\theta\ e^{ik\theta}\hat{f}(\sinh^2\theta) \tag{3.10}$$

then by the convolution theorem the transformed integral relation (3.8) becomes

$$\tilde{\hat{f}}(k) = \tilde{\hat{f}}_1(k)\tilde{\hat{f}}_2(k)\,. \tag{3.11}$$

Thus we may solve integrals of the general type given in (3.1) by this double integral transform method provided that it is possible to make the transforms $f_i(\xi)\to\hat{f}_i(\rho)\to\tilde{\hat{f}}_i(k)$ for both the functions f_1 and f_2 *and* that the subsequent inverse transforms of the resulting function $\tilde{\hat{f}}(k)$ can be made. Of course the form of the functions f_1 and f_2 are crucial in order for this procedure to be successfully undertaken. For the typical cases which arise in the diagrammatic expansion of a conformal field theory this method has proven to be very successful, although the intermediate steps often involve non-trivial manipulations of hypergeometric functions. In the next section several examples which are likely to occur in calculations in conformal field theory are given to illustrate the method, and provide a table of transforms for future reference.

4. Illustration of the method

For application to the calculation of two-point functions in a conformal field theory we may use this method to solve the integrals over products of propagators that occur in a diagrammatic expansion of the theory. Therefore, by considering, for example, the Green function of the free scalar field given in (2.8) we wish to solve integrals of the following type:

$$I = \int_0^\infty dz \int d^{d-1}r\ \frac{1}{(2z)^\beta}\frac{1}{\left(\tilde{s}^2\right)^\alpha\left(\bar{\tilde{s}}^2\right)^{\overline{\alpha}}\left(\tilde{s}'^2\right)^{\alpha'}\left(\bar{\tilde{s}}'^2\right)^{\overline{\alpha}'}} \tag{4.1}$$

with

$$\tilde{s}^2 = (x-r)^2 + (y-z)^2 \qquad \bar{\tilde{s}}^2 = (x-r)^2 + (y+z)^2$$

$$\tilde{s}'^2 = (x'-r)^2 + (y'-z)^2 \qquad \bar{\tilde{s}}'^2 = (x'-r)^2 + (y'+z)^2 .$$

For conformal invariance, following (2.5), we must also require

$$\alpha + \overline{\alpha} + \alpha' + \overline{\alpha}' + \beta = d . \tag{4.2}$$

This integral may be readily cast into the general form (3.1), for which we should then take

$$f_1(\tilde{\xi}) = \frac{1}{(2y)^{\alpha+\overline{\alpha}}} \frac{1}{\tilde{\xi}^{\alpha}(1+\tilde{\xi})^{\overline{\alpha}}} \qquad f_2(\tilde{\xi}') = \frac{1}{(2y')^{\alpha'+\overline{\alpha}'}} \frac{1}{\tilde{\xi}'^{\alpha'}(1+\xi')^{\overline{\alpha}'}} . \tag{4.3}$$

Later in this section we will consider the more general integrals that arise in the discussion of the large-N expansion of the $O(N)$ sigma model, where the propagator for the auxiliary field λ has a more complicated functional form.

To solve the integral (4.1) using the method of section 3 we first take the sequence of transforms $f \to \hat{f} \to \hat{\hat{f}}$ as defined in (3.3) and (3.10) for functions of the form $f_i(\xi)$ above. For simplicity we take

$$f(\xi) = \frac{1}{\xi^{\alpha}(1+\xi)^{\overline{\alpha}}} \tag{4.4}$$

The first transform $f \to \hat{f}$ follows from standard references:

$$\begin{aligned}\hat{f}(\rho) &= \frac{\pi^{\lambda}}{\Gamma(\lambda)} \int_0^{\infty} \mathrm{d}u \; u^{\lambda-1} \frac{1}{(u+\rho)^{\alpha}(1+u+\rho)^{\overline{\alpha}}} \\ &= \pi^{\lambda} \frac{\Gamma(\alpha+\overline{\alpha}-\lambda)}{\Gamma(\alpha+\overline{\alpha})} \frac{1}{(1+\rho)^{\alpha+\overline{\alpha}-\lambda}} F\left(\alpha+\overline{\alpha}-\lambda, \alpha\,;\, \alpha+\overline{\alpha}\,;\, \frac{1}{1+\rho}\right) .\end{aligned} \tag{4.5}$$

The function $F(a, b; c; z)$ is a hypergeometric function whose definition is given in (A.1) For the subsequent transform $\hat{f} \to \tilde{f}$ we consider the cases $\overline{\alpha} = 0$, $\alpha = 0$, $\alpha = \overline{\alpha}$ separately

$$\begin{aligned} f_{\mathrm{I}}(\xi) &= \frac{1}{\xi^{\alpha}} \\ \hat{f}_{\mathrm{I}}(\rho) &= \pi^{\lambda} \frac{\Gamma(\alpha-\lambda)}{\Gamma(\alpha)} \frac{1}{\rho^{\alpha-\lambda}} \\ \tilde{\hat{f}}_{\mathrm{I}}(k) &= \pi^{\frac{1}{2}d} \frac{\Gamma(2\alpha-2\lambda)\Gamma(1-2\alpha+2\lambda)}{\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha-\lambda)} \left[\frac{\Gamma(\alpha-\lambda+\frac{1}{2}\mathrm{i}k)}{\Gamma(1-\alpha+\lambda+\frac{1}{2}\mathrm{i}k)} + \frac{\Gamma(\alpha-\lambda-\frac{1}{2}\mathrm{i}k)}{\Gamma(1-\alpha+\lambda-\frac{1}{2}\mathrm{i}k)} \right] \end{aligned} \tag{4.6}$$

$$\begin{aligned} f_{\mathrm{II}}(\xi) &= \frac{1}{(1+\xi)^{\overline{\alpha}}} \\ \hat{f}_{\mathrm{II}}(\rho) &= \pi^{\lambda} \frac{\Gamma(\overline{\alpha}-\lambda)}{\Gamma(\overline{\alpha})} \frac{1}{(1+\rho)^{\overline{\alpha}-\lambda}} \\ \tilde{\hat{f}}_{\mathrm{II}}(k) &= \pi^{\frac{1}{2}d} \frac{1}{\Gamma(\overline{\alpha})\Gamma(\frac{1}{2}+\overline{\alpha}-\lambda)} \Gamma(\overline{\alpha}-\lambda+\tfrac{1}{2}\mathrm{i}k)\Gamma(\overline{\alpha}-\lambda-\tfrac{1}{2}\mathrm{i}k) \end{aligned} \tag{4.7}$$

$$f_{\rm III}(\xi) = \frac{1}{\xi^\alpha(1+\xi)^\alpha}$$

$$\hat{f}_{\rm III}(\rho) = \pi^\lambda \frac{\Gamma(2\alpha-\lambda)}{\Gamma(2\alpha)} \frac{1}{(1+\rho)^{2\alpha-\lambda}} F\left(2\alpha-\lambda, \alpha\,;\, 2\alpha\,;\, \frac{1}{1+\rho}\right) \tag{4.8}$$

$$\tilde{\hat{f}}_{\rm III}(k) = \pi^{\frac{1}{2}d} 4^{\alpha-\frac{1}{2}d} \frac{\Gamma(\frac{1}{2}d-\alpha)}{\Gamma(\alpha)} \frac{\Gamma(\alpha-\frac{1}{2}\lambda-\frac{1}{4}ik)\Gamma(\alpha-\frac{1}{2}\lambda+\frac{1}{4}ik)}{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda-\frac{1}{4}ik)\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+\frac{1}{4}ik)}$$

There is one other case, a particular combination of two functions of the type (4.4), which is of interest, namely

$$f_{\rm IV}(\xi) = \frac{2\xi+1}{\xi^\alpha(1+\xi)^\alpha}$$

$$\hat{f}_{\rm IV}(\rho) = 2\pi^\lambda \frac{\Gamma(2\alpha-\lambda-1)}{\Gamma(2\alpha-1)} \frac{1}{(1+\rho)^{2\alpha-\lambda-1}} F\left(2\alpha-\lambda-1, \alpha-1\,;\, 2\alpha-2\,;\, \frac{1}{1+\rho}\right) \tag{4.9}$$

$$\tilde{\hat{f}}_{\rm IV}(k) = \pi^{\frac{1}{2}d} 4^{\alpha-\frac{1}{2}d} \frac{\Gamma(\frac{1}{2}d-\alpha)}{\Gamma(\alpha)} \frac{\Gamma(\alpha-\frac{1}{2}\lambda-\frac{1}{2}-\frac{1}{4}ik)\Gamma(\alpha-\frac{1}{2}\lambda-\frac{1}{2}+\frac{1}{4}ik)}{\Gamma(\frac{1}{2}\lambda-\frac{1}{4}ik)\Gamma(\frac{1}{2}\lambda+\frac{1}{4}ik)}.$$

The last two cases, $f_{\rm III}$ and $f_{\rm IV}$, are important because the more general case where $\overline{\alpha}$ differs from α by any integer follows in a straightforward manner from them. However, the derivation of those two results directly is non-trivial. The simplest way to verify them is by working backwards and taking the inverse transforms. A general procedure for taking the inverse transforms is discussed next.

For application to conformal field theory where we have integrals of the form (3.1) then the transformed relation (3.11) suggests that we need to take the inverse transform of products of the functions $\hat{f}_\iota(k)$ in I to IV. In all of these cases the dependence of $\hat{f}(k)$ on k is through combinations of Gamma functions. Consequently, by considering the poles of the Gamma function, the inverse Fourier transform $\tilde{\hat{f}} \to \hat{f}$ of (3.11) can be performed by contour integration. We first consider the following combination of Gamma functions which is appropriate for verifying the transforms of $f_{\rm III}$ and $f_{\rm IV}$ above:

$$\tilde{\hat{g}}_{a,b}(k) \equiv \frac{\Gamma(a-\frac{1}{4}ik)\Gamma(a+\frac{1}{4}ik)}{\Gamma(b-\frac{1}{4}ik)\Gamma(b+\frac{1}{4}ik)}. \tag{4.10}$$

The poles of $\Gamma(a-\frac{1}{4}ik)$ occur at $\frac{1}{4}ik = a+n$ with residue $(-1)^n/n!$ (for n a non-negative integer). Therefore, the inverse transform is obtained as a sum of the residues of $\tilde{\hat{g}}_{a,b}$, resulting in a series that has hypergeometric form

$$\begin{aligned}\hat{g}_{a,b}(\sinh^2\theta) &= \frac{1}{2\pi}\int {\rm d}k\, {\rm e}^{-ik\theta}\, \tilde{\hat{g}}_{a,b}(k) \\ &= \frac{4\Gamma(2a)}{\Gamma(b-a)\Gamma(b+a)}\, {\rm e}^{-4a|\theta|} F\big(2a, a-b+1; a+b; {\rm e}^{-4|\theta|}\big) \\ &= \frac{4\Gamma(2a)}{\Gamma(b-a)\Gamma(b+a)} \frac{1}{(4\cosh^2\theta)^{2a}} F\left(2a, a+b-\frac{1}{2}; 2a+2b-1; \frac{1}{\cosh^2\theta}\right).\end{aligned} \tag{4.11}$$

By choosing appropriate values for a, b, and noting that $\cosh^2\theta = 1+\rho$ then the Fourier transformed functions $\hat{f}_{\rm III}$ and $\hat{f}_{\rm IV}$ follow directly from this result. To obtain the inverse parallel transform we use

$$\frac{1}{\Gamma(-\lambda)}\int_0^\infty \mathrm{d}\rho\ \rho^{-\lambda-1}\frac{1}{(1+\rho+\xi)^p}=\frac{\Gamma(p+\lambda)}{\Gamma(p)}\frac{1}{(1+\xi)^{p+\lambda}} \tag{4.12}$$

with $p=2a+n$, in the last line of (4.11) so that

$$g_{a,b}(\xi)=\frac{\Gamma(2a+\lambda)}{4^{2a-1}\pi^\lambda\Gamma(b-a)\Gamma(b+a)}\frac{1}{(1+\xi)^{2a+\lambda}}$$
$$\times F\left(2a+\lambda,a+b-\frac{1}{2};2a+2b-1;\frac{1}{1+\xi}\right). \tag{4.13}$$

Now, with the appropriate choice of a,b, we can use this result to verify the parallel transforms $\hat{f}_{\rm III}$ and $\hat{f}_{\rm IV}$ in equations (4.8) and (4.9).

In order to solve the integrals of the type (3.1) we must find the inverse Fourier transform of products of the functions $\hat{f}_i(k)$ in I to IV. These may can be simply obtained as hypergeometric series by contour integration in a similar way to above above calculation. The procedure for finding the inverse parallel transform differs, though, because it is not always possible to make the simplifying manipulation of the hypergeometric function that is made in (4.11). This is because the hypergeometric series is often of higher order. However, a procedure for taking the inverse transform $\hat{f}\to f$ which bypasses this step is derived in the appendix. This procedure makes essential use of a special property of the hypergeometric series which arises on taking the inverse Fourier transform, that is due the symmetry $\tilde{\hat{g}}(k)=\tilde{\hat{g}}(-k)$. After taking the inverse Fourier transform of products of the functions in I to IV, we obtain a hypergeometric series with one of the two following forms:

$$\hat{g}(\sinh^2\theta)=\mathrm{e}^{-4a|\theta|}{}_{q+1}F_q\big(2a,b_1,\cdots b_q;c_1,\cdots c_q;\mathrm{e}^{-4|\theta|}\big) \tag{4.14}$$

$$\hat{h}(\sinh^2\theta)=\mathrm{e}^{-2a|\theta|}{}_{q+1}F_q\big(2a,b_1,\cdots b_q;c_1,\cdots c_q;\mathrm{e}^{-2|\theta|}\big) \tag{4.15}$$

where the notation ${}_{q+1}F_q$ refers to a generalized hypergeometric series which is defined in (A.2). The crucial point is that the parameters b_i and c_i in these functions are always related by $c_i=1+2a-b_i$.

We now present the inverse transforms of six of the possible combinations of the functions in I to IV, which have been obtained using this method. These represent solutions to particular integrals of the type (3.1). First we consider products of the functions $f_{\rm I}$ and $f_{\rm II}$. In these cases the inverse Fourier transform results in hypergeometric series of the form (4.15) and the inverse parallel transform can be found via the methods outlined in the appendix. Thus, using (A.22), we obtain

$$\mathcal{I}_{\rm I,I}(\xi)=\int_0^\infty \mathrm{d}z\int \mathrm{d}^{d-1}r\ \frac{1}{(2z)^d}\frac{1}{\xi^\alpha}\frac{1}{\xi'^{\alpha'}}$$
$$=\pi^{\frac{1}{2}d}\frac{\Gamma(1+\alpha+\alpha'-d)\Gamma(\frac{1}{2}d-\alpha-\alpha')}{\Gamma(1-\frac{1}{2}d)\Gamma(\frac{1}{2}d)}F\big(\alpha,\alpha';1+\alpha+\alpha'-\tfrac{1}{2}d;-\xi\big)$$
$$+\pi^{\frac{1}{2}d}\frac{\Gamma(\alpha+\alpha-\frac{1}{2}d)\Gamma(\frac{1}{2}d-\alpha)\Gamma(\frac{1}{2}d-\alpha')}{\Gamma(d-\alpha-\alpha')\Gamma(\alpha)\Gamma(\alpha')}\frac{1}{\xi^{\alpha+\alpha'-\frac{1}{2}d}}$$
$$\times F\big(\tfrac{1}{2}d-\alpha,\tfrac{1}{2}d-\alpha',1+\tfrac{1}{2}d-\alpha-\alpha';-\xi\big) \tag{4.16}$$

$$\mathcal{I}_{\rm I,II}(\xi) = \int_0^\infty dz \int d^{d-1}r \, \frac{1}{(2z)^d} \frac{1}{\tilde{\xi}^\alpha} \frac{1}{(1+\tilde{\xi}')^{\alpha'}}$$

$$= \pi^{\frac{1}{2}d} \frac{\Gamma(1+\alpha+\alpha'-d)\Gamma(\frac{1}{2}d-\alpha)}{\Gamma(\frac{1}{2}d)\Gamma(1+\alpha'-\frac{1}{2}d)} F(\alpha, \alpha'; \tfrac{1}{2}d; -\xi) \tag{4.17}$$

$$\mathcal{I}_{\rm II,II}(\xi) = \int_0^\infty dz \int d^{d-1}r \, \frac{1}{(2z)^d} \frac{1}{(1+\tilde{\xi})^\alpha} \frac{1}{(1+\tilde{\xi}')^{\alpha'}}$$

$$= \pi^{\frac{1}{2}d} \frac{\Gamma(1+\alpha+\alpha'-d)}{\Gamma(1+\alpha+\alpha'-\frac{1}{2}d)} F(\alpha, \alpha'; 1+\alpha+\alpha'-\tfrac{1}{2}d; -\xi) . \tag{4.18}$$

In order to bring these results to this form it is necessary to use several identities of the hypergeometric function which can be found in the standard references [9].

If we take the limit $\alpha + \alpha' \to d$ in these integrals, which corresponds to $\beta \to 0$ in the original integral (4.1) then the following relation:

$$\frac{1}{\left((x-x')^2\right)^{\frac{1}{2}d-\beta}} \sim \frac{1}{2\beta} S_d \delta^d(x-x') \qquad \text{as } \beta \to 0 \tag{4.19}$$

can be used to show that

$$\mathcal{I}_{\rm I,I} + \mathcal{I}_{\rm II,II} = \pi^d \frac{\Gamma(\frac{1}{2}d-\alpha)\Gamma(\frac{1}{2}d-\alpha')}{\Gamma(\alpha)\Gamma(\alpha')} \delta^d(x-x') \tag{4.20}$$

in the limit $\alpha + \alpha' \to d$. This is the expected result when the range of the integral (4.1), with $\overline{\alpha} = \overline{\alpha}' = \beta = 0$, is extended to the infinite space $\mathbb{R}^d$. In a similar way it is possible to show that if $\alpha + \alpha' = d$ then $\mathcal{I}_{\rm I,II} + \mathcal{I}_{\rm II,I} = 0$, where $\mathcal{I}_{\rm II,I}$ is defined by taking $\alpha \leftrightarrow \alpha'$ in $\mathcal{I}_{\rm I,II}$.

We now evaluate three more conformally invariant integrals involving combinations of the functions $f_{\rm III}$ and $f_{\rm IV}$. In these cases the inverse Fourier transform results in a hypergeometric series of the form (4.14). One obtains

$$\mathcal{I}_{\rm III,III}(\xi) = \int_0^\infty dz \int d^{d-1}r \, \frac{1}{(2z)^d} \frac{1}{\tilde{\xi}^\alpha(1+\tilde{\xi})^\alpha} \frac{1}{\tilde{\xi}'^{\alpha'}(1+\tilde{\xi}')^{\alpha'}}$$

$$= \frac{\Gamma(\frac{1}{2}d-\alpha')\Gamma(\alpha'-\alpha)\Gamma(\alpha+\alpha'-\lambda)}{\Gamma(\frac{1}{2}d-\alpha)\Gamma(\alpha')\Gamma(\alpha+\frac{1}{2})} \pi^{\frac{1}{2}d} 4^{\alpha'-\frac{1}{2}d} \frac{1}{[\xi(1+\xi)]^\alpha}$$

$$\times {}_3F_2\left(\alpha, 1+\alpha-\frac{1}{2}d, \frac{1}{2}d-\alpha'; \alpha+\frac{1}{2}, 1+\alpha-\alpha'; -\frac{1}{4\xi(1+\xi)}\right)$$

$$+\alpha \leftrightarrow \alpha' \tag{4.21}$$

$$\mathcal{I}_{\rm III,IV}(\xi) = \int_0^\infty dz \int d^{d-1}r \, \frac{1}{(2z)^d} \frac{1}{\tilde{\xi}^\alpha(1+\tilde{\xi})^\alpha} \frac{2\tilde{\xi}'+1}{\tilde{\xi}'^{\alpha'}(1+\tilde{\xi}')^{\alpha'}}$$

$$= \frac{\Gamma(\frac{1}{2}d-\alpha)\Gamma(\frac{1}{2}d-\alpha')\Gamma(\alpha+\alpha'-\frac{1}{2}d)}{\Gamma(\alpha)\Gamma(\alpha')\Gamma(d-\alpha-\alpha')} \pi^{\frac{1}{2}d} \frac{1}{[\xi(1+\xi)]^{\alpha+\alpha'-\frac{1}{2}d}}$$

$$\times F\left(\lambda-\alpha, \tfrac{1}{2}d-\alpha'; d-\alpha-\alpha'; -4\xi(1+\xi)\right) \tag{4.22}$$

$$\mathcal{I}_{\rm IV,IV}(\xi) = \int_0^\infty dz \int d^{d-1}r \, \frac{1}{(2z)^d} \frac{2\tilde{\xi}+1}{\tilde{\xi}^\alpha(1+\tilde{\xi})^\alpha} \frac{2\tilde{\xi}'+1}{\tilde{\xi}'^{\alpha'}(1+\tilde{\xi}')^{\alpha'}}$$

$$= \frac{\Gamma(\frac{1}{2}d-\alpha')\Gamma(\alpha'-\alpha)\Gamma(\alpha+\alpha'-1-\lambda)}{\Gamma(\frac{1}{2}d-\alpha)\Gamma(\alpha')\Gamma(\alpha-\frac{1}{2})}\pi^{\frac{1}{2}d}4^{\alpha'-\frac{1}{2}d}\frac{2\xi+1}{[\xi(1+\xi)]^{\alpha}}$$
$$\times {}_3F_2\left(\alpha, 1+\alpha-\frac{1}{2}d, \frac{1}{2}d-\alpha'; \alpha-\frac{1}{2}, 1+\alpha-\alpha'; -\frac{1}{4\xi(1+\xi)}\right)$$
$$+\alpha \leftrightarrow \alpha' . \tag{4.23}$$

To solve for $\mathcal{I}_{\mathrm{III,III}}$ we require the transformed function $\tilde{\hat{f}}_{\mathrm{III}}(k)$ with the result for the inverse transform of the general case (A.11) which is given in the appendix. For $\mathcal{I}_{\mathrm{IV,IV}}$ we use $\hat{f}_{\mathrm{IV}}(k)$ with the inverse transform (A.17). To obtain $\mathcal{I}_{\mathrm{III,IV}}$ in the form (4.22), we follow a similar procedure to the other two cases, but also use a relationship between hypergeometric functions with argument $-z$ and hypergeometric functions with argument $-1/z$ to simplify the expression.

The solution to the integrals in (4.16) to (4.23) all have a pole at $\alpha = d/2$ except for (4.18). This pole arises due to the short distance logarithmic singularity for $r \sim x$ in each of these integrals when $\alpha = d/2$.

We are now in a position to evaluate integrals of the type (3.1) with products of more general functions than those discussed thus far. For example, if we consider the function $g_{a,b}$ given in (4.13) which was derived from the definition of $\tilde{\hat{g}}_{a,b}$ in (4.10), then since

$$\tilde{\hat{g}}_{a,b}(k)\tilde{\hat{g}}_{b,c}(k) = \tilde{\hat{g}}_{a,c}(k) \tag{4.24}$$

it follows directly that

$$\int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}r\, \frac{1}{(2z)^d} g_{a,b}(\tilde{\xi}) g_{b,c}(\tilde{\xi}') = \begin{cases} g_{a,c}(\xi) & a \neq c \\ (4yy')^{\frac{1}{2}d}\delta^d(x-x') & a = c\,. \end{cases} \tag{4.25}$$

This is a solution to an integral of the product of two hypergeometric functions with the special form (4.13). This relation is useful in the large-N expansion of the $O(N)$ sigma model with the Ordinary transition, where the Green function of the auxiliary field λ is a hypergeometric function of exactly this type [7, 10, 11].

We may generalize this further by considering the function

$$\tilde{\hat{g}}_{ab,c\delta}(k) \equiv \frac{\Gamma(a-\frac{1}{4}\mathrm{i}k)\Gamma(a+\frac{1}{4}\mathrm{i}k)\Gamma(b-\frac{1}{4}\mathrm{i}k)\Gamma(b+\frac{1}{4}\mathrm{i}k)}{\Gamma(c-\frac{1}{4}\mathrm{i}k)\Gamma(c+\frac{1}{4}\mathrm{i}k)\Gamma(\delta-\frac{1}{4}\mathrm{i}k)\Gamma(\delta+\frac{1}{4}\mathrm{i}k)}. \tag{4.26}$$

The methods of the appendix can then be used to obtain the inverse transforms of this function provided $\delta = \frac{1}{2}\lambda$ or $\delta = \frac{1}{2} + \frac{1}{2}\lambda$. The inverse Fourier transform gives†

$$\hat{g}_{ab,c\delta}(\sinh^2\theta) = \frac{4\Gamma(2a)\Gamma(b-a)\Gamma(b+a)}{\Gamma(c-a)\Gamma(c+a)\Gamma(\delta-a)\Gamma(\delta+a)}\,\mathrm{e}^{-4a|\theta|}$$
$$\times {}_4F_3\left(2a, b+a, 1+a-c, 1+a-\delta; 1+a-b, c+a, \delta+a\,; \mathrm{e}^{-4|\theta|}\right)$$
$$+a \leftrightarrow b\,. \tag{4.27}$$

Subsequently, using (A.11) for the case $\delta = \frac{1}{2} + \frac{1}{2}\lambda$ we find

$$g_{ab,c\delta}(\xi) = \frac{1}{4^{2a-1}\pi^\lambda}\frac{\Gamma(2a+\lambda)\Gamma(b-a)\Gamma(b+a)}{\Gamma(c-a)\Gamma(c+a)\Gamma(\frac{1}{2}+\frac{1}{2}\lambda-a)\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+a)}\frac{1}{[\xi(1+\xi)]^{a+\frac{1}{2}\lambda}}$$

† This function, is related to the Meijer's G-function which is defined by the contour integration of combinations of Gamma functions with arguments of a particular form [9].

$$\times {}_3F_2\left(a+\tfrac{1}{2}\lambda, \tfrac{1}{2}+a-\tfrac{1}{2}\lambda, c-b\,; 1+a-b, a+c\,; -\frac{1}{4\xi(1+\xi)}\right)$$
$$+a \leftrightarrow b \tag{4.28}$$

whereas when $\delta = \frac{1}{2}\lambda$, using (A.17) we obtain

$$g_{ab,c\delta}(\xi) = \frac{1}{4^{2a-1}\pi^\lambda} \frac{\Gamma(2a+\lambda)\Gamma(b-a)\Gamma(b+a)}{\Gamma(c-a)\Gamma(c+a)\Gamma(\frac{1}{2}\lambda-a)\Gamma(\frac{1}{2}\lambda+a)} \frac{\xi+\frac{1}{2}}{[\xi(1+\xi)]^{\frac{1}{2}+a+\frac{1}{2}\lambda}}$$
$$\times {}_3F_2\left(\frac{1}{2}+a+\frac{1}{2}\lambda, 1+a-\frac{1}{2}\lambda, c-b\,; 1+a-b, a+c\,; -\frac{1}{4\xi(1+\xi)}\right)$$
$$+a \leftrightarrow b\,. \tag{4.29}$$

Thus provided δ is one of $\frac{1}{2}\lambda$ or $\frac{1}{2}+\frac{1}{2}\lambda$ then $g_{ab,c\delta}(\xi)$ can be obtained as ${}_3F_2$ hypergeometric functions. The solutions to the integrals in (4.21)–(4.23) represent special cases of these functions. More generally, integrals of products of these types of ${}_3F_2$ hypergeometric functions are possible. Since

$$\tilde{\tilde{g}}(k)_{ab,c\delta}\, \tilde{\tilde{g}}_{ce,bf}(k) = \tilde{\tilde{g}}(k)_{ae,\delta f} \tag{4.30}$$

then it follows that

$$\int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}r\, \frac{1}{(2z)^d} g_{ab,c\delta}(\tilde{\xi}) g_{ce,bf}(\tilde{\xi}') = g_{ae,\delta f}(\xi) \tag{4.31}$$

provided $\delta, f = \frac{1}{2}\lambda, \frac{1}{2}+\frac{1}{2}\lambda$. Similar integral relations can be derived by considering possible combinations of $g_{a,b}$ with $g_{ab,c\delta}$ with particular choices of the parameters a, b, c, δ. Integrals such as these occur in a discussion of the large-N expansion of the $O(N)$ sigma model with the Special transition where the Green function for the auxiliary field λ contains hypergeometric functions of this type [7, 11, 12].

5. Integrals involving spin factors

We now turn our attention to conformally invariant integrals involving spin factors which occur in the discussion of two-point functions of vector and tensor fields. For this we define fields X_μ, $\tilde{X}_\mu$, with scale dimension zero, which transform as vectors at the point x under conformal transformations that leave the boundary fixed.

$$X_\mu = \frac{y}{\xi^{\frac{1}{2}}(1+\xi)^{\frac{1}{2}}}\partial_\mu \xi \qquad \tilde{X}_\mu = \frac{y}{\tilde{\xi}^{\frac{1}{2}}(1+\tilde{\xi})^{\frac{1}{2}}}\partial_\mu \tilde{\xi}\,. \tag{5.1}$$

These are constructed to be unit vectors so that $X_\mu X_\mu = \tilde{X}_\mu \tilde{X}_\mu = 1$.

We will use the example of an integral with one spin factor in the integrand to illustrate the method. Such an integral would be appropriate for correlation functions involving a single vector field. We define

$$I_\mu = I(\xi)X_\mu = \int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}r\, \frac{1}{(2z)^d} \tilde{X}_\mu f_1(\tilde{\xi}) f_2(\tilde{\xi}') \tag{5.2}$$

which has the functional form $I_\mu = I(\xi)X_\mu$ due to conformal invariance. To find $I(\xi)$ we use the fact that X_μ is a unit vector to obtain

$$I(\xi) = \int_0^\infty \mathrm{d}z \int \mathrm{d}^{d-1}r\, \frac{1}{(2z)^d} (X\cdot\tilde{X}) f_1(\tilde{\xi}) f_2(\tilde{\xi}')\,. \tag{5.3}$$

Now, since

$$X \cdot \tilde{X} = \frac{(2\xi+1)(2\tilde{\xi}+1) - (2\tilde{\xi}'+1)}{4\big(\xi(1+\xi)\tilde{\xi}(1+\tilde{\xi})\big)^{\frac{1}{2}}} \tag{5.4}$$

the methods of section 4 can be used to solve for $I(\xi)$ in terms of hypergeometric functions. For example, if we take

$$f_1(\xi) = \frac{1}{\xi^{\alpha}(1+\xi)^{\alpha}} \qquad f_2(\xi) = \frac{1}{\xi^{\alpha'}(1+\xi)^{\alpha'}} \tag{5.5}$$

then we may use the solution to the integral $\mathcal{I}_{\rm III,IV}$ in (4.22) to obtain

$$I(\xi) = \pi^{\frac{1}{2}d} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}d-\alpha)\Gamma(\frac{1}{2}d-\alpha')\Gamma(\frac{1}{2}+\alpha+\alpha'-\frac{1}{2}d)}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\alpha')\Gamma(\frac{1}{2}+d-\alpha-\alpha')} \frac{1}{[\xi(1+\xi)]^{\alpha+\alpha'-\frac{1}{2}d}}$$
$$\times F\left(d-2\alpha, d-2\alpha'; \tfrac{1}{2}+d-\alpha-\alpha'; -\xi\right). \tag{5.6}$$

The case where $\alpha = \frac{1}{2}(d-1)$ can be worked out by an alternative method by noting that†

$$\partial_\mu \left(\frac{1}{(2y)^{d-1}} \frac{\tilde{X}_\mu}{[\tilde{\xi}(1+\tilde{\xi})]^{\frac{1}{2}(d-1)}} \right) = S_d \delta^d(x-r) \tag{5.7}$$

so that from the definition of I_μ in (5.2),

$$\partial_\mu \left(\frac{1}{(2y)^{d-1}} I_\mu \right) = S_d \frac{1}{(2y)^d} \frac{1}{[\xi(1+\xi)]^{\alpha'}} . \tag{5.8}$$

This result may be rewritten as a differential equation for $I(\xi)$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left([\xi(1+\xi)]^{\frac{1}{2}(d-1)} I(\xi) \right) = \tfrac{1}{2} S_d \frac{1}{[\xi(1+\xi)]^{1+\alpha'-\frac{1}{2}d}} \tag{5.9}$$

which may be solved to give

$$I(\xi) = \frac{S_d}{d-2\alpha'} \frac{1}{[\xi(1+\xi)]^{\alpha'-\frac{1}{2}}} F\left(1, d-2\alpha'; 1+\tfrac{1}{2}d-\alpha'; -\xi\right). \tag{5.10}$$

The constant of integration is taken to be zero, because otherwise the presence of such a term would violate (5.8) by producing an extra delta function contribution to the RHS. This solution is in agreement with (5.6). A similar procedure can be used for integrals involving more spin factors, which would be appropriate for correlation function involving the energy momentum tensor or two vector fields for example. Such integrals are evaluated in I by a slightly different method.

6. Large-N expansion for the $O(N)$ model

In this section, by way of conclusion, we demonstrate the use of the of the parallel transform method to to calculate two-point functions in the $1/N$ expansion of the $O(N)$ non-linear sigma model for the case of semi-infinite geometry. As usual, the nonlinear constraint on the fields $\phi_\alpha(x)$; $\phi^2 = N$ can be removed by introducing an auxiliary field $\lambda(x)$ in the Lagrangian via an interaction term $\mathcal{L}_1 = \frac{1}{2}\lambda\phi^2$. To analyse the two-point functions of the fields ϕ_α and λ we first define

$$\langle \phi_\alpha(x)\phi_\beta(x')\rangle = G_\phi(x,x')\delta_{\alpha\beta} \qquad \langle \lambda(x)\lambda(x')\rangle = G_\lambda(x,x'). \tag{6.1}$$

† For this we recall that $-\partial^2 s^{2-d} = (d-2)S_d\delta^d(s)$.

Then, to zeroth order in the $1/N$ expansion, these Green functions satisfy the following relations [13]:

$$\left(-\nabla^2 + \langle\lambda(x)\rangle\right) G_\phi(x,x') = \delta^d(x-x') \tag{6.2}$$

$$\int \mathrm{d}^d r\, G_\phi^2(x,r) G_\lambda(r,x') = -\frac{2}{N}\delta^d(x-x'). \tag{6.3}$$

Both of these relations may be solved by making use of conformal invariance and using the parallel transform method discussed in section 3. For this we write

$$G_\phi(x,x') = \frac{1}{(4yy')^{\eta_\phi}} f_\phi(\xi) \qquad G_\lambda(x,x') = \frac{1}{(4yy')^{\eta_\lambda}} f_\lambda(\xi). \tag{6.4}$$

Since $2\eta_\phi + \eta_\lambda = d$ due to conformal invariance of the integral in (6.3), then the zeroth order result $\eta_\phi = \frac{1}{2}d - 1$ implies that $\eta_\lambda = 2$ to this order. Now, with the scaling relation $\langle\lambda(x)\rangle = A_\lambda/4y^2$, it is possible to obtain G_ϕ as a solution to a differential equation. Alternatively we can recast (6.2) into an integral equation so that the method of parallel transforms can be used to obtain a solution. Writing

$$\int \mathrm{d}^d r\, H(x,r) G_\phi(r,x') = \delta^d(x-x') \tag{6.5}$$

requires that

$$H(x,x') = \left(-\nabla^2 + \frac{A_\lambda}{4y^2}\right)\delta^d(x-x'). \tag{6.6}$$

The integral of $H(x,x')$ over planes parallel to the boundary may be written as

$$\int \mathrm{d}^{d-1}\boldsymbol{x}\, H(x,x') = \frac{1}{(4yy')^{\frac{3}{2}}}\hat{h}(y,y') \tag{6.7}$$

defining $\hat{h}$ to be

$$\hat{h}(\mathrm{e}^{2\theta},\mathrm{e}^{2\theta'}) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + 1 + A_\lambda\right)\delta(\theta-\theta'). \tag{6.8}$$

The subsequent Fourier transform of $\hat{h}(\mathrm{e}^{2\theta},\mathrm{e}^{2\theta'})$ gives the simple expression

$$\tilde{\hat{h}}(k) = k^2 + 1 + A_\lambda. \tag{6.9}$$

We may now solve for G_ϕ by first integrating the integral equation (6.5) over planes parallel to the boundary and then taking the Fourier transform as defined in (3.10). The resulting equation is

$$\tilde{\hat{h}}(k)\tilde{\hat{f}}_\phi(k) = 1 \tag{6.10}$$

where $\tilde{\hat{f}}_\phi(k)$ is the transform of the function $f_\phi(\xi)$ defined in (6.4). Consequently the desired result is

$$\tilde{\hat{f}}_\phi(k) = \frac{1}{\tilde{\hat{h}}(k)} = \frac{1}{k^2+1+A_\lambda}. \tag{6.11}$$

If we are now express $\tilde{\hat{f}}_\phi(k)$ as

$$\tilde{\hat{f}}_\phi(k) = \frac{1}{16}\frac{\Gamma(\mu+\frac{1}{4}\mathrm{i}k)\Gamma(\mu-\frac{1}{4}\mathrm{i}k)}{\Gamma(1+\mu+\frac{1}{4}\mathrm{i}k)\Gamma(1+\mu-\frac{1}{4}\mathrm{i}k)} \qquad \mu^2 = \frac{1+A_\lambda}{16} \tag{6.12}$$

then we may use the result (4.13) to obtain the inverse transform directly:

$$f_\phi(\xi) = \frac{1}{4^{1+2\mu}\pi^\lambda} \frac{\Gamma(2\mu+\lambda)}{\Gamma(1+2\mu)} \frac{1}{(1+\xi)^{2\mu+\lambda}} F\left(2\mu+\lambda, \frac{1}{2}+2\mu; 1+4\mu; \frac{1}{1+\xi}\right). \tag{6.13}$$

This general form for $f_\phi(\xi)$ gives the correct large-N Green function $G_\phi(x, x')$ appropriate for both the ordinary and special transitions in the statistical mechanical context where we should take $\mu = (d-3)/4$ and $\mu = (d-5)/4$ respectively [14]. Solutions for $G_\lambda(x, x')$ can now be obtained in a similar way via the integral equation (6.3). Results for $G_\lambda(x, x')$ for both the Ordinary and Special transitions were calculated with the parallel transform method in I and also in [10–12] by a different method. It would be interesting to see if the next order in the $1/N$ expansion can be obtained using the methods discussed in this paper; this is the subject of future research.

Acknowledgments

I wish to thank Hugh Osborn for many useful ideas and suggestions. This research was funded by a Postdoctoral Research Fellowship from the National Science and Engineering Research Council of Canada.

Appendix. Hypergeometric function relations

In this appendix we derive some essential hypergeometric function relations that are needed in section 4. We start with the definition of the hypergeometric function

$$F(a, b; c; z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a)_n (b)_n}{(c)_n} \tag{A.1}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. There is a natural generalization of this definition, which is called a generalized hypergeometric series

$${}_pF_q(a_1, \cdots a_p; c_1, \cdots c_q; z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n}. \tag{A.2}$$

For application to section 4 we need to consider the inverse parallel transform of functions of the following hypergeometric form:

$$\hat{g}(\sinh^2\theta) = \mathrm{e}^{-4a|\theta|}\, {}_{q+1}F_q\big(2a, b_1, \cdots b_q; c_1, \cdots c_q; \mathrm{e}^{-4|\theta|}\big) \tag{A.3}$$

$$\hat{h}(\sinh^2\theta) = \mathrm{e}^{-2a|\theta|}\, {}_{q+1}F_q\big(2a, b_1, \cdots b_q; c_1, \cdots c_q; \mathrm{e}^{-2|\theta|}\big) \tag{A.4}$$

where $c_i = 1 + 2a - b_i$. To take the inverse transform we express these results as the sum indicated by (A.2) and then observe that, with $\rho = \sinh^2\theta$

$$\mathrm{e}^{-2p|\theta|} = \left(\sqrt{\rho} + \sqrt{1+\rho}\right)^{-2p} = \frac{1}{4^p(1+\rho)^p} F\left(p, \frac{1}{2}+p; 1+2p; \frac{1}{1+\rho}\right) \tag{A.5}$$

where $p = 2a + 2n$ for $\hat{g}$ and $p = a + n$ for $\hat{h}$. We can now obtain the inverse transform by using

$$\frac{1}{\Gamma(-\lambda)} \int_0^\infty \mathrm{d}\rho\, \rho^{-\lambda-1} \frac{1}{(1+\rho+\xi)^p} = \frac{\Gamma(p+\lambda)}{\Gamma(p)} \frac{1}{(1+\xi)^{p+\lambda}} \tag{A.6}$$

with the result

$$\frac{\Gamma(p)}{\Gamma(p+\lambda)\Gamma(-\lambda)}\int_0^\infty \mathrm{d}\rho\, \rho^{-\lambda-1}\frac{1}{(1+\rho+\xi)^p}F\left(p,\frac{1}{2}+p\,;1+2p\,;\frac{1}{1+\rho+\xi}\right)$$
$$=\frac{1}{(1+\xi)^{p+\lambda}}F\left(p+\lambda,\frac{1}{2}+p\,;1+2p\,;\frac{1}{1+\xi}\right)$$
$$=\frac{1}{\xi^{p+\lambda}}F\left(p+\lambda,\frac{1}{2}+p\,;1+2p\,;-\frac{1}{\xi}\right) \tag{A.7}$$
$$=\frac{\xi+\frac{1}{2}}{\left[\xi(1+\xi)\right]^{\frac{1}{2}(p+\lambda+1)}}F\left(\frac{1}{2}(p+\lambda+1),1+\frac{1}{2}(p-\lambda);1+p\,;-\frac{1}{4\xi(1+\xi)}\right) \tag{A.8}$$
$$=\frac{1}{\left[\xi(1+\xi)\right]^{\frac{1}{2}(p+\lambda)}}F\left(\frac{1}{2}(p+\lambda),\frac{1}{2}(1+p-\lambda);1+p\,;-\frac{1}{4\xi(1+\xi)}\right). \tag{A.9}$$

The techniques for finding the inverse transform of $\hat{g}$ and $\hat{h}$ are related, except that for $\hat{h}$ it is necessary to use (A.7) while for $\hat{g}$ either of the equivalent results (A.8),(A.9) may be used. Both of these equivalent results are helpful, because two different expressions for $g(\xi)$ can be derived from them and a nice simplification of these expressions occurs for different values of the parameters b_i. For the purpose of this discussion we will focus on the inversion of $\hat{g}$ for which we use (A.9) with $p = 2a + 2n$ to obtain

$$g(\xi)=\frac{1}{4^{2a}\pi^\lambda}\frac{\Gamma(2a+\lambda)}{\Gamma(2a)}\frac{1}{\left[\xi(1+\xi)\right]^{a+\frac{1}{2}\lambda}}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(2a)_n(b_1)_n\cdots(b_q)_n}{(c_1)_n\cdots(c_q)_n}\frac{(2a+\lambda)_{2n}}{4^n(2a)_{2n}}$$
$$\times\frac{1}{\left[4\xi(1+\xi)\right]^n}F\left(a+\frac{1}{2}\lambda+n,\frac{1}{2}+a-\frac{1}{2}\lambda+n\,;\right.$$
$$\left.1+2a+2n\,;-\frac{1}{4\xi(1+\xi)}\right). \tag{A.10}$$

By expanding the hypergeometric function, $g(\xi)$ can be rewritten as

$$g(\xi)=\frac{1}{4^{2a}\pi^\lambda}\frac{\Gamma(2a+\lambda)}{\Gamma(2a)}\frac{1}{\left[\xi(1+\xi)\right]^{a+\frac{1}{2}\lambda}}\sum_{N=0}^{\infty}G_N\frac{(-1)^N}{[4\xi(1+\xi)]^N} \tag{A.11}$$

where the coefficient G_N is given by the finite sum

$$G_N=\sum_{n=0}^{N}\left(\frac{(-1)^n}{n!(N-n)!}\frac{(2a)_n(b_1)_n\cdots(b_q)_n}{(c_1)_n\cdots(c_q)_n}\right.$$
$$\left.\times\frac{(2a+\lambda)_{2n}(a+\frac{1}{2}\lambda+n)_{N-n}(\frac{1}{2}+a-\frac{1}{2}\lambda+n)_{N-n}}{4^n(2a)_{2n}(1+2a+2n)_{N-n}}\right). \tag{A.12}$$

This can be simplified further by using the following identities for the Pochhammer symbol:

$$(p)_{2n}=4^n(\tfrac{1}{2}p)_n(\tfrac{1}{2}+\tfrac{1}{2}p)_n \qquad (p+n)_{N-n}=\frac{(p)_N}{(p)_n} \qquad (p)_{-n}=\frac{(-1)^n}{(1-p)_n}. \tag{A.13}$$

so that G_N becomes

$$G_N = \frac{(a+\frac{1}{2}\lambda)_N(\frac{1}{2}+a-\frac{1}{2}\lambda)_N}{N!(1+2a)_N}\,{}_{q+4}F_{q+3}\Big(2a, 1+a, b_1, \cdots, b_q, \tfrac{1}{2}+a+\tfrac{1}{2}\lambda, -N;$$
$$a, c_1, \cdots, c_q, \tfrac{1}{2}+a-\tfrac{1}{2}\lambda, 1+2a+N; 1\Big). \tag{A.14}$$

The finite ${}_{q+4}F_{q+3}$ finite hypergeometric series, with argument 1, has a special form because $c_i = 1+2a-b_i$. As a consequence, for the particular case $q=1$, the resulting ${}_5F_4$ can be summed exactly by a special limit of Dougall's theorem which states that [15]

$${}_5F_4(2a, 1+a, b, c, -N; a, 1+2a-b, 1+2a-c, 1+2a+N; 1)$$
$$= \frac{(1+2a)_N(1+2a-b-c)_N}{(1+2a-b)_N(1+2a-c)_N}. \tag{A.15}$$

So for $q=1$ and applying Dougall's theorem we find

$$g(\xi) = \frac{1}{4^{2a}\pi^\lambda}\frac{\Gamma(2a+\lambda)}{\Gamma(2a)}\frac{1}{[\xi(\xi+1)]^{a+\frac{1}{2}\lambda}}$$
$$\times {}_2F_1\left(a+\frac{1}{2}\lambda, \frac{1}{2}-\frac{1}{2}\lambda+a-b; 1+2a-b; -\frac{1}{4\xi(1+\xi)}\right). \tag{A.16}$$

This result and the inverse fourier transform (4.11) are sufficient for verifying the integral transforms in (4.8) and (4.9).

Although a generalization of Dougall's theorem is not known for arbitrary q, the coefficient G_N can usually be simplified to give a finite ${}_5F_4$ series through cancellation of the parameters. However, for this cancellation to occur it may be necessary in some cases to use an alternative formula for $g(\xi)$ which may be derived from (A.8) following a similar procedure. The result is

$$g(\xi) = \frac{1}{4^{2a}\pi^\lambda}\frac{\Gamma(2a+\lambda)}{\Gamma(2a)}\frac{\xi+\frac{1}{2}}{[\xi(1+\xi)]^{\frac{1}{2}+a+\frac{1}{2}\lambda}}\sum_{N=0}^{\infty}\overline{G}_N\frac{(-1)^N}{[4\xi(1+\xi)]^N} \tag{A.17}$$

where

$$\overline{G}_N = \frac{(\frac{1}{2}+a+\frac{1}{2}\lambda)_N(1+a-\frac{1}{2}\lambda)_N}{N!(1+2a)_N}\,{}_{q+4}F_{q+3}\Big(2a, 1+a, b_1, \cdots, b_q, a+\tfrac{1}{2}\lambda, -N;$$
$$a, c_1, \cdots, c_q, 1+a-\tfrac{1}{2}\lambda, 1+2a+N; 1\Big). \tag{A.18}$$

The calculation of $h(\xi)$ proceeds in a similar way. Using (A.7) one gets

$$h(\xi) = \frac{1}{4^a\pi^\lambda}\frac{\Gamma(a+\lambda)}{\Gamma(a)}\frac{1}{\xi^{a+\lambda}}\sum_{N=0}^{\infty}H_N\frac{(-1)^N}{\xi^N} \tag{A.19}$$

where

$$H_N = \frac{(a+\lambda)_N(\frac{1}{2}+a)_N}{N!(1+2a)_N}\,{}_{q+3}F_{q+2}\Big(2a, 1+a, b_1, \cdots, b_q, -N;$$
$$a, c_1, \cdots, c_q, 1+2a+N; 1\Big). \tag{A.20}$$

Again we recall that $c_i = 1 + 2a - b_i$. For the case $q = 1$ which is relevant for section 4 we use a theorem similar to Dougall's:

$$ {}_4F_3\,(2a, 1+a, b, -N; a, 1+2a-b, 1+2a+N; 1) = \frac{(1+2a)_N(\frac{1}{2}+a-b)_N}{(\frac{1}{2}+a)_N(1+2a-b)_N} \qquad \text{(A.21)} $$

to obtain

$$ h(\xi) = \frac{1}{4^a\pi^\lambda}\frac{\Gamma(a+\lambda)}{\Gamma(a)}\frac{1}{\xi^{a+\lambda}}F\left(a+\lambda, \frac{1}{2}+a-b; 1+2a-b\,; -\frac{1}{\xi}\right). \qquad \text{(A.22)} $$

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